Iterative Processes for Solving Incorrect Convex Variational Problems*

A. KAPLAN and R. TICHATSCHKE
Fachbereich Mathematik, University Rostock, 0-2500 Rostock, Universitätsplatz 1, Germany.

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Abstract. The present paper is concerned with a general approach to the construction and the numerical analysis of stable methods solving semi-infinite convex programs and variational inequalities of elliptical type in case where the considered problems are incorrect. The approach which is based on the application of the PROX-regularization (cf. Martinet, 1970; Ekeland and Temam, 1976; Rockafellar, 1976; Brézis and Lions, 1978; Lemaire, 1988) secures the strong convergence of the minimizing sequence. The possibility of the algorithmical realization is described and depends on the smoothness properties of the solutions.

Key words. Regularization, weakly coercive variational inequalities, semi-infinite programming problems, numerical algorithms for convex programming problems.

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1. Introduction

The problem under consideration is the following:

$$J(u) \rightarrow \inf, \quad u \in K_0, \tag{1}$$

where $J: V \rightarrow \mathbb{R}$ is a convex and continuous functional, K_0 is a convex and closed subset of a Hilbert space V. Moreover, it is assumed that

$$U^0 := \{u \in K_0 : J(u) = \inf\{J(v) : v \in K_0\}\} \neq \emptyset .$$

The solution of a problem of such a type will usually be obtained by applying an approximation of the set K_0 by K_i and of the functional J by J_i.

If J is strongly convex and Gateaux-differentiable then the following statement about the convergence of a sequence $\{u^i\}$, $u^i \approx \arg\min_{u \in K_i} J(u)$, to $u^{*,0} = \arg\min_{u \in K_0} J(u)$ takes place (for different variants of this proposition cf. Mosco, 1969; Pankov, 1979). Denote $u^{*,i} = \arg\min_{u \in K_i} J(u)$ and suppose that there are given convex, closed sets K_i and the gradient criterion

$$\|\nabla J(u^i) - \nabla J(u^{*,i})\|_{V^*} \leq \varepsilon_i \quad (i = 1, 2, \ldots)$$

is fulfilled with $\lim \varepsilon_i = 0$ (V^* is the topological dual space to V); furthermore, we consider the sets

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$$S_r(v) = \{u \in V: \|u - v\| \leq r\},$$
$$\tilde{Q}_i = K_i \cap S_r(u^{*,0}), \quad i = 0, 1, 2, \ldots,$$

and ρ_H denotes the Hausdorff distance.

THEOREM 1. *For fixed $r > 0$ it is assumed that* $\lim \rho_H(\tilde{Q}_0, \tilde{Q}_i) = 0$. *Then for the iterates strong convergence to the optimal solution of the initial problem* (1) *holds, i.e.,* $\lim u^i = u^{*,0}$.

Theorem 1 is a corollary of Theorem A by Mosco (1969) and characterizes the well-posedness of the problem.

But if the functional J is not strongly convex, problem (1) can be ill-posed and usually we can only prove that

$$\lim J(u^i) = \inf\{J(u): u \in K_0\}.$$

For such a class of ill-posed problems a general scheme is described below constructing a sequence $\{u^i\}$, $\lim \min_{v \in K_i} \|u^i - v\| = 0$, which converges strongly to $u^{*,0} \in U^0 = \text{Arg}\min_{u \in K_0} J(u)$, if $U^0 \neq \emptyset$.

2. The Iterative Process and Its Convergence

Let $\{J_i\}$ $(i = 1, 2, \ldots)$ be a family of convex and Gateaux-differentiable functionals such that

$$\sup_{u \in V} |J(u) - J_i(u)| \leq \sigma_i, \quad \lim \sigma_i = 0$$

and $\{K_i\}$ $(i = 1, 2, \ldots)$, $K_i \subset V$ be a family of convex, closed sets approximizing K_0.

For a fixed u^0 and a given sequence $\{\varepsilon_i\}$ $\{i = 1, 2, \ldots)$ the following formal iterative procedure is considered in order to construct a minimizing sequence $\{u^i\}$ for the initial problem (1):

$$\|\nabla\chi_i(u^i) - \nabla\chi_i(\bar{u}^i)\|_{V^*} \leq \varepsilon_i, \tag{2}$$

with the regularized functional

$$\chi_i(u) = J_i(u) + \|u - u^{i-1}\|^2, \tag{3}$$

where $\bar{u}^i = \arg\min\{\chi_i(u): u \in K_i\}$, $\varepsilon_i \geq 0$, $\lim \varepsilon_i = 0$ and $\|.\|$ denotes the norm of the elements in V.

The properties of the PROX-mapping

$$\text{Prox}_{G,f}a: a \to \arg\min_{u \in G} \{f(u) + \|u - a\|^2\}$$

using for this approach are described, for instance, by Ekeland and Temam (1976), except Lemma 2 below, which is new in our opinion. Therefore, we make the following notations:

$$\rho(P, T) = \sup_{v \in P} \rho(v, T), \quad \rho(v, T) = \inf_{w \in T} \|v - w\|,$$

$$\left.\begin{aligned} &Q_i = K_i \cap S_r(0), \\ &\psi_i(u) = J(u) + \|u - u^{i-1}\|^2, \\ &q^i = \arg\min_{u \in Q_0} \psi_i(u), \end{aligned}\right\} \quad (i = 0, 1, 2, \ldots)$$

$\hat{Q} = K' \cap S_r(0)$, where K' is a subset of K_0 containing U^0 and $\{q^i\}$.

Suppose that

$$\sup_{u,u' \in S_r(0)} \frac{|J(u) - J(u')|}{\|u - u'\|} \leq \nu(r) < \infty.$$

ASSUMPTION 1. *Given a continuous increasing functional* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(0) = 0$, *such that the estimates*

$$\rho(Q_i, Q_0) \leq \varphi(h_i) \qquad (4)$$

$$(i = 0, 1, 2, \ldots)$$

$$\rho(\hat{Q}, Q_i) \leq \varphi(h_i) \qquad (5)$$

hold true for a fixed sequence $\{h_i\}$ *with* $h_i \geq h_{i+1}$ *and* $\lim h_i = 0$.

ASSUMPTION 2. *For some* $r > 0$ *and for each* $i \geq 1$ *there exists a point* $\hat{u}^i \in K_i \cap S_r(0)$ *such that the inequality*

$$J(\hat{u}^i) \leq \inf_{u \in K_i} J(u) + 2\nu(r)\delta_i, \quad (i = 0, 1, 2, \ldots) \qquad (6)$$

is satisfied with $\lim \delta_i = 0$ (*in general it is not assumed that* $\operatorname{Arg\,min}_{u \in K_i} J(u) \neq \emptyset$).

For further investigations, if method (2), (3) is considered, it is assumed that the choice of r corresponds with Assumption 2.

We shall specially seperate the case if the convex functional J satisfies the following condition, which is weaker than the usual assumption about the strong convexity for the energy functionals: in particular, in this case the initial problem can have more than one solution or no solutions.

CONDITION (*). The inequality

$$J(u) - J(v) \geq \langle j(u), u - v \rangle + \delta \|Pu - Pv\|^2$$

holds true for some $\delta > 0$ and each $u, v \in V$, where $\langle .\,, . \rangle$ denotes the duality connection between V and V^* and P is an orthogonal projector on a subspace $V' \subset V$ of finite defect and $j(v)$ is an arbitrary element of the subdifferential $\partial J(v)$.

Condition $(*)$ is natural for elliptical variational inequalities and, obviously, it is always satisfied for convex semi-infinite problems.

THEOREM 2. *Suppose the Assumptions 1 and 2 are fulfilled, and for some* $r' \leq r$ *let* $U^0 \cap S_{r'/8}(0) \neq \emptyset$; $u^0 \in K_0 \cap S_{r'/4}(0)$ *and*

$$\sum_{i=1}^{\infty}\left[(2\nu(r)(\varphi(h_i)+\delta_i)+2\sigma_i)^{1/2}+2\varphi(h_i)+\frac{\varepsilon_i}{2}\right]<\frac{r'}{2}\,. \tag{7}$$

Then, using the iteration process (2), (3), *the estimates* $\|u^i\| \leq r'$ *and* $\|\bar{u}^i\| \leq r'$ *are true, where* $\bar{u}^i = \arg\min_{u\in K_i} \chi_i(u)$. *If, moreover,* $\sum_{i=1}^{\infty} \varepsilon_i^{1/2} < \infty$, *then* $\{u^i\}$ *weakly converges to some* $u^* \in U^0$, *and subject to the additional conditions* $(*)$ *the relation* $\lim_{i\to\infty} \|u^i - u^*\| = 0$ *holds.*

The proof of this main result essentially uses the following statements, which are of interest by themselves, too. Consider $\theta_i(u) = J(u) + \|u - \xi^{i-1}\|^2$, where ξ^0 is arbitrarily chosen and the sequence $\{\xi^i\}$ let be satisfy the conditions

$$\theta_i(\xi^i) \leq \min_{u\in Q_0} \theta_i(u) + \gamma_i\,, \quad \rho(\xi^i, Q_0) \leq \bar{\gamma}_i \quad (i = 0, 1, 2, \ldots)\,. \tag{8}$$

LEMMA 1. *Suppose* $U^0 \neq \emptyset$ *and the parameters* γ_i *and* $\bar{\gamma}_i$ *in* (8) *are chosen such that* $\sum_{i=1}^{\infty} \gamma_i^{1/2} < \infty$ *and* $\sum_{i=1}^{\infty} \bar{\gamma}_i^{1/2} < \infty$. *Then the sequence* $\{\xi^i\}$ *weakly converges to some element* ξ^* *of the set* U^0 *and, if* J *satisfies the condition* $(*)$, *the relation* $\lim_{i\to\infty} \|\xi^i - \xi^*\| = 0$ *is true.*

Proof. Denote $\bar{\gamma}_0 = \sup_{i=1,2,\ldots} \bar{\gamma}_i$. Since $\xi^{i-1} \in S_{r+\bar{\gamma}_0}(0)$ for each i and due to the estimate

$$\begin{aligned}&|(u-\xi^{i-1}, u-\xi^{i-1}) - (u'-\xi^{i-1}, u'-\xi^{i-1})|\\ &\quad \leq 4(r+\bar{\gamma}_0)\|u-u'\| \text{ for any } u, u' \in S_{r+\bar{\gamma}_0}(0)\end{aligned}$$

together with the Lipschitz property of J, we obtain $|\theta_i(u) - \theta_i(u')| \leq [\nu(r+\bar{\gamma}_0) + 4(r+\bar{\gamma}_0)]\|u-u'\| = \tilde{\nu}(r)\|u-u'\|$, where $\tilde{\nu}(r) = \nu(r+\bar{\gamma}_0) + 4(r+\bar{\gamma}_0)$. Hence, with $\hat{\xi}^i \equiv \arg\min_{w\in Q_0} \|\xi^i - w\|$, one gets $|\theta_i(\hat{\xi}^i) - \theta_i(\xi^i)| \leq \tilde{\nu}(r)\bar{\gamma}_i$ and, due to the choice of ξ^i in (8),

$$\theta_i(\hat{\xi}^i) \leq \min_{u\in Q_0} \theta_i(u) + \gamma_i + \tilde{\nu}(r)\bar{\gamma}_i = \min_{u\in Q_0} \theta_i(u) + \tilde{\gamma}_i\,,$$

where $\tilde{\gamma}_i = \gamma_i + \tilde{\nu}(r)\bar{\gamma}_i$.

Because of the strong convexity of θ_i for $\bar{\xi}^i \equiv \arg\min_{u\in Q_0} \theta_i(u)$ the inequalities

$$\begin{aligned}&\theta_i(\hat{\xi}^i) - \theta_i(\bar{\xi}^i) \geq \langle \tau_i(\bar{\xi}^i), \hat{\xi}^i - \bar{\xi}^i\rangle + \|\hat{\xi}^i - \bar{\xi}^i\|^2\,,\\ &\langle \tau_i(\bar{\xi}^i), \hat{\xi}^i - \bar{\xi}^i\rangle \geq 0\end{aligned}$$

are true for some $\tau_i(\bar{\xi}^i) \in \partial\theta_i(\bar{\xi}^i)$ and we can estimate

$$\|\hat{\xi}^i - \bar{\xi}^i\| \leq \tilde{\gamma}_i^{1/2} .$$

Therefore,

$$\|\bar{\xi}^i - \xi^i\| \leq \tilde{\gamma}_i^{1/2} + \bar{\gamma}_i . \tag{9}$$

Because of (8), (9) the Theorem 1 in Rockafellar (1976) guarantees that $\{\xi^i\}$ weakly converges to some element $u^* \in U^0$ and $\lim_{i\to\infty} \|\xi^i - \xi^{i-1}\| = 0$. Hence

$$\lim_{i\to\infty} \|\bar{\xi}^i - \xi^{i-1}\| = 0 . \tag{10}$$

Now, let the condition $(*)$ be fulfilled. Regarding the definition of the points $\bar{\xi}^i$ and the Proposition 2.2.2 in Ekeland/Temam (1976), we get

$$J(v) \geq J(\bar{\xi}^i) + 2\langle \xi^{i-1} - \bar{\xi}^i), v - \bar{\xi}^i \rangle \quad \text{for all } v \in Q_0 \tag{11}$$

and since J is a weak lower semi-continuous functional one can conclude from (10), (11) that

$$\lim_{i\to\infty} J(\bar{\xi}^i) = J(u^*) . \tag{12}$$

But due to (8)

$$J(\xi^i) \leq |\bar{\xi}^i - \xi^{i-1}\|^2 + J(\bar{\xi}^i) + \gamma_i \tag{13}$$

and using (10), (12) and (13) we obtain

$$J(u^*) \leq \liminf_{i\to\infty} J(\xi^i) \leq \limsup_{i\to\infty} J(\xi^i) \leq J(u^*) .$$

Consequently,

$$\lim_{i\to\infty} J(\xi^i) = J(u^*) . \tag{14}$$

The relation $\lim_{i\to\infty} \|\xi^i - u^*\| = 0$ follows now obviously from condition $(*)$ and (14). ■

LEMMA 2. *Let $f: V \to \mathbb{R}$ be a convex, continuous functional and $G \subset V$ be a convex, closed set; furthermore,*

$$f_{\min} = \inf_{u\in G} f(u) > -\infty , \quad f(\bar{p}) - f_{\min} \leq \delta \text{ for some } \bar{p} \in G . \tag{15}$$

Then the estimate

$$\|\text{Prox}_{G,f} a^0 - \bar{p}\| \leq \|a^0 - \bar{p}\| + \sqrt{\delta}$$

holds true for each $a^0 \in V$.

Proof. Due to the determination $a^1 \equiv \text{Prox}_{G,f} a^0$ and the theorem on the subdifferential over the sum of functionals (cf. Ekeland/Temam, 1976) there exists an element ζ of the subdifferential $\partial f(a^1)$ such that

$$\langle \zeta + 2I(a^1 - a^0), u - a^1 \rangle \geq 0 \quad \text{for each } u \in G . \tag{16}$$

Because of the convexity of f and of inequality (15) we get $\langle \zeta, \bar{p} - a^1 \rangle \leq \delta$, which together with (16) leads to

$$\langle 2I(a^1 - a^0), -\bar{p} + a^1 \rangle \leq \delta \,.$$

Therefore,

$$\|a^1 - \bar{p}\|^2 - \|a^0 - \bar{p}\|^2 \leq -(a^1 - a^0, a^1 - a^0) + 2(a^1 - a^0, a^1 - \bar{p})$$
$$\leq -\|a^1 - a^0\|^2 + \delta \,,$$

$$\|a^1 - \bar{p}\| \leq \|a^0 - \bar{p}\| + \sqrt{\delta} \,. \quad \blacksquare$$

The result of this lemma can be trivially extended to the case, when $f: V \to \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{+\infty\}$ is a convex, lower semi-continuous functional with $\inf_{u \in V} f(u) > -\infty$ and $\operatorname{dom} f \neq \emptyset$.

Together with these facts we are able now to prove Theorem 2.

Proof of Theorem 2. Let $u^* \in U^0 \cap S_{r'/8}(0)$. Due to the conditions (4), (5) the points $v^i \in Q_i$, and $\bar{v}^i \in Q_0$ can be chosen such that

$$\|v^i - u^*\| \leq \varphi(h_i) \,, \quad \|\bar{v}^i - \hat{u}^i\| \leq \varphi(h_i) \,, \quad i = 1, 2, \ldots \tag{17}$$

Hence $J(\bar{v}^i) - J(\hat{u}^i) \leq \nu(r)\varphi(h_i)$. Taking into consideration that $J(u^*) \leq J(\bar{v}^i)$ and property (6) hold true, we conclude

$$J(u^*) - \inf_{u \in K_i} J(u) \leq \nu(r)(\varphi(h_i) + 2\delta_i) \,,$$

which together with $J(u^*) \geq J(v^i) - \nu(r)\varphi(h_i)$ leads to

$$J(v^i) - \inf_{u \in K_i} J(u) \leq 2\nu(r)(\varphi(h_i) + \delta_i)$$

and hence

$$J_i(v^i) - \inf_{u \in K_i} J_i(u) \leq 2\nu(r)(\varphi(h_i) + \delta_i) + 2\sigma_i \equiv \mu_i \,.$$

Using now Lemma 2 with $G = K_i$, $a^0 = u^{i-1}$, $\bar{p} = v^i$, $f = J_i$ and $\delta = \mu_i$, we estimate

$$\|\bar{u}^i - v^i\| \leq \|u^{i-1} - v^i\| + \sqrt{\mu_i} \,.$$

Regarding the strong convexity of χ_i, the choice of $\bar{u}^i$ and the condition (2), the following inequalities hold:

$$\|\bar{u}^i - u^*\| \leq \|u^{i-1} - u^*\| + \sqrt{\mu_i} + 2\varphi(h_i) \,, \tag{18}$$

$$\|u^i - u^*\| \leq \|u^{i-1} - u^*\| + \sqrt{\mu_i} + 2\varphi(h_i) + \frac{\varepsilon_i}{2} \,. \tag{19}$$

From the inequalities (7), (18), (19) and the choice of u^0 we get

$$\|u^1\| \leq r' \quad \text{and} \quad \|\bar{u}^1\| \leq r' \,.$$

Summarizing the inequalities in (19) for $i = 1, 2, \dots, k$ it follows

$$\|u^k - u^*\| \leq \|u^0 - u^*\| + \sum_{i=1}^{k} \left(\sqrt{\mu_i} + 2\varphi(h_i) + \frac{\varepsilon_i}{2} \right)$$

and on account of (19) (for $i = k + 1$) we have

$$\max\{\|\bar{u}^{k+1} - u^*\|, \|u^{k+1} - u^*\|\}$$
$$\leq \|u^0 - u^*\| + \sum_{i=1}^{k+1} \left(\sqrt{\mu_i} + 2\varphi(h_i) + \frac{\varepsilon_i}{2} \right),$$

therefore, $\|u^{k+1}\| \leq r'$ and $\|\bar{u}^{k+1}\| \leq r'$ can be concluded. If $\psi_i(\bar{u}^i) \geq \psi_i(q^i)$, we choose $z^i \in Q_i$ such that $\|z^i - q^i\| \leq \varphi(h_i)$. Hence, $|\psi(z^i) - \psi_i(q^i)| \leq (\nu(r) + 4r)\varphi(h_i)$ and together with $\chi_i(z^i) \geq \chi_i(\bar{u}^i)$ the inequalities

$$\psi_i(z^i) \geq \psi_i(\bar{u}^i) - 2\sigma_i \geq \psi_i(q^i) - 2\sigma_i ,$$
$$\psi_i(\bar{u}^i) - \psi_i(q^i) \leq (\nu(r) + 4r)\varphi(h_i) + 2\sigma_i$$

are true.

Taking into consideration the strong convexity of χ_i, condition (2) and

$$\sup_{u,u' \in S_r(0)} \frac{|J(u) - J(u')|}{\|u - u'\|} \leq \nu(r) < \infty ,$$

we get the estimate

$$\psi_i(u^i) - \psi_i(\bar{u}^i) \leq (\nu(r) + 4r) \frac{\varepsilon_i}{2} ,$$

this implies

$$\psi_i(u^i) - \psi_i(q^i) \leq (\nu(r) + 4r)\left(\varphi(h_i) + \frac{\varepsilon_i}{2} \right) + 2\sigma_i .$$

In case of $\psi_i(\bar{u}^i) < \psi_i(q^i)$ it is clear that

$$\psi_i(u^i) - \psi_i(q^i) < (\nu(r) + 4r) \frac{\varepsilon_i}{2} .$$

Besides, $\rho(u^i, Q_0) \leq \varphi(h_i) + \varepsilon_i/2$ holds. Now to finish the proof we can use Lemma 1. ■

REMARK 1. Lemaire (1988, Theorem 3) (see also Theorem 3.2 in Alart and Lemaire, 1991) has been established the convergence of an analogous iterative process under the conditions that J, $J_i : V \rightarrow \bar{\mathbb{R}}$ are proper, closed and convex functionals, $K = K_i = V$ and the closeness between J and J_i is measured in terms of their Moreau–Yosida approximations. This requires to provide a sufficient exactness of the approximation for any element of the set Q_0 by elements of the set Q_i (see the last condition of Theorem 3 in Lemaire, 1988).

However, as a rule, this is impossible, if finite element approximations are applied for variational inequalities, because in this case the elements of Q_0 are not sufficiently smooth (see Theorem 3 below and Hlavacek *et al.*, 1986).

REMARK 2. If J is a convex, Gateaux-differentiable functional and the closeness between J and J_i is defined by

$$\sup_{u \in V} \|\nabla J(u) - \nabla J_i(u)\|_{V^*} \leq \sigma_i ,$$

then the statement of Theorem 2 (replacing inequality (7) by

$$\sum_{i=1}^{\infty} \left[(2\varphi(r))^{1/2}(\varphi(h_i) + \delta_i)^{1/2} + 2\varphi(h_i) + 2\sigma_i + \frac{\varepsilon_i}{2} \right] < \frac{r'}{2})$$

holds true without the assumption that J_i is a convex functional (cf. Kaplan/ Tichatschke, 1991). From the practical point of view this is essential, because the errors of the approximation can transform the incorrect problem into a global optimization problem.

REMARK 3. Analyzing the described scheme, it should be noted that, on principle, the parameters $\{h_i\}$, $\{\sigma_i\}$ and $\{\varepsilon_i\}$ can be chosen freely, i.e., it can be guaranteed the necessary order for the approximation of the set Q_0, of the functional J and the required exactness of the solution in the approximated problems. Hereby the conditions of Theorem 2 are not contradictory, if the choice of $\{K_i\}$ takes place with a quick convergence of $\{\delta_i\}$ to 0. If the choice of the parameters r and $\{\delta_i\}$, corresponding to Assumption 2 and inequality (7), cannot be guaranteed (cf. Example 2 below), then the iteration procedure (2), (3) has to be modified by exchanging J_i with

$$\tilde{J}_i(u) = J_i(u) + \rho^2(u, S_{r_1}(0)) , \tag{20}$$

where $r_1 > 0$ is chosen such that $U^0 \cap S_{r_1}(0) \neq \emptyset$.

For $\tilde{J}(u) = J(u) + \rho^2(u, S_{r_1}(0))$ the Assumption 2 is satisfied for sufficiently large r with $\delta_i \equiv 0$ $(i = 1, 2, \ldots)$, and the assertions of Theorem 2 are valid. This process is developed by Kaplan (1990) for the case that J is a quadratical functional.

3. Application to Variational Inequalities and SIP

It is clear that the realization of method (2), (3) depends on the possibility to construct a functional φ with the required properties and can be studied using the considerated approach for solving concrete problems.

Now the described scheme will be briefly sketched for solving elliptical variational inequalities, where for the construction of the family $\{K_i\}$ the method of finite elements is used on a sequence of triangulations with the characteristic parameter $h_i \to 0$ and where the inclusion $K_i \subset K_0$ is guaranteed for all i. The determination of function φ is connected with serious difficulties, which as a rule, result from the weak smoothness of the solutions of variational inequalities. Now problem (1) is considered with

$$J(u) = \frac{1}{2}\langle Au, u\rangle - \langle f, u\rangle ,$$

where $A: V \to V^*$ is an elliptical operator of second kind, $V = H^1$, $f \in H^0 = L_2(\Omega)^m$ (H^i are Sobolev spaces of m-dimensional vector functions on Ω), and Ω is a plane and connected domain with sufficiently smooth boundary Γ.

Suppose that the solutions of the initial problem and of the problem min $\{J(u) + \|u\|^2 : u \in K_0\}$ belong to H^2 for every $f \in H^0$ and there exists a constant c_0 independently of f and u, for which the following inequality is true:

$$\|u\|_2^2 \leq c_0(\|f\|_0^2 + \|u\|_1^2) , \tag{21}$$

where $\|.\|_2$, $\|.\|_0$ and $\|.\|_1$ are norms in the corresponding spaces H^2, H^0 and H^1.

Now we are able to analyse the choice of function φ for the process (2), (3) starting from (21). For the sake of simplicity we consider the case $J_i = J$.

Suppose the triangulation sequence $\{\Omega_{h_i}\}$ of the domain Ω fulfils the standard properties of regularity (cf. for instance Glowinski *et al.*, 1981). Then we have the estimate

$$\|v - v_h\| \leq c\|v\|_2 h \tag{22}$$

if $v \in H^2$ is interpolated by a linear combination v_h of piece-wise linear basis functions, where c is independent of v. This estimate is also true at Ω as well as at the grid domains Ω_h for $\Omega_h \supset \Omega$ (by a corresponding continuation of the function v).

Let $f \in H^0$ be fixed and

$$\varphi(h) = (c_0[\|f\|_0 + 2r]^2 + r^2)^{1/2} ch \equiv \hat{c}h$$

and, moreover, $r > 4\|f\|_{V^*}$, $U^0 \cap S_{r'/8}(0) \neq \emptyset$, where

$$r' < \frac{r}{2(M+4)}\left(\left(1 + r^{-1}\left(\frac{r}{4} - \|f\|_{V^*}\right)\right)^{1/2} - 1\right) \tag{23}$$

$$\left(M \geq \sup_{\|v\| \neq \emptyset} \frac{\langle Av, v\rangle}{\|v\|^2}\right).$$

We note that in the proof of the first statement of Theorem 2 instead of condition (5) only the condition $\rho(u^*, Q_i) \leq \varphi(h_i)$ ($i = 1, 2, \ldots$) is used, where $u^* \in U^0 \cap S_{r'/8}(0)$. Because of (21) the inequality

$$\|u^*\|_2^2 \leq c_0\left(\|f\|_0^2 + \left(\frac{r'}{8}\right)^2\right)$$

can be obtained, therefore

$$\|u^* - u^*_{h_i}\| \leq \hat{c}h_i = \varphi(h_i) \quad (i = 1, 2, \ldots) ,$$

and if $\varphi(h_i) < \frac{r}{2}$, we get $u^*_{h_i} \in Q_i$ such that

$$\rho(u^*, Q_i) \leq \|u^* - u^*_h\| \leq \varphi(h_i) .$$

In this way, if the assumptions of Theorem 2 are fulfilled with the function $\varphi(h) = \hat{c}h$, the estimates $\|u^i\| \leq r'$ and $\|\bar{u}^i\| \leq r'$ are true for each $i = 1, 2, \ldots$. Due to the inequality

$$J(u) + \|u - u^{i-1}\|^2 \leq \left(\frac{M}{2} + 1\right)\|u\|^2 + \|f\|_{V^*}\|u\| + \|u^{i-1}\|^2 + 2|(u, u^{i-1})|$$

and

$$u^* \in Q_0 \cap S_{r'/8}(0) , \quad \text{if } \|u^{i-1}\| \leq r' ,$$

we can estimate

$$\min_{u \in Q_0} \{J(u) + \|u - u^{i-1}\|^2\} < (M + 4)(r')^2 + \|f\|_{V^*} r' . \tag{24}$$

On the other hand, for $\|u\| > r/2$ the inequality

$$J(u) + \|u - u^{i-1}\|^2 \geq \frac{r^2}{4} - r'r - \frac{1}{2} \|f\|_{V^*} r \tag{25}$$

holds true. According to (24), (25) one can conclude that if

$$(M + 4)(r')^2 + \|f\|_{V^*} r' < \frac{r^2}{4} - r'r - \frac{1}{2} \|f\|_{V^*} r \tag{26}$$

the points q^i belong to $S_{r/2}(0)$.

Since $r' < r/2$, condition (26) is obviously satisfied if

$$(M + 4)(r')^2 + r'r < \frac{r^2}{4} - \|f\|_{V^*} r ,$$

but this inequality is a conclusion of (23).

On this way we obtain $q^i \in S_{r/2}(0)$ and, therefore, $q^i = \arg\min_{u \in K_0} \psi_i(u)$. Using the expression

$$\psi_i(u) = \frac{1}{2} \langle Au, u \rangle + \|u\|^2 - 2(u, u^{i-1}) - \langle f, u \rangle + \|u^{i-1}\|^2 ,$$

from inequality (21) it follows

$$\|q^i\|_2 \leq (c_0(\|f\|_0 + 2r')^2 + (r')^2)^{1/2}$$

and together with (22), if $\varphi(h_i) < r/2$ holds, there exists an element $z^i \in Q_i$ such that $\|z^i - q^i\| \leq \varphi(h_i)$.

Finally, if for $\varphi(h) = \hat{c}h$ the choice of $\{h_i\}$, $\{\delta_i\}$ and $\{\varepsilon_i\}$ satisfies the condition (7), then estimate (5) is fulfilled with the described function φ. This leads to the statement of

THEOREM 3. *Let* $\varphi(h) = [c_0(\|f\|_0 + 2r)^2 + r^2]^{1/2}ch = \hat{c}h$ *and moreover,* $r > 4\|f\|_{V^*}$ *and* r' *satisfy condition* (23) *and the conditions of Theorem* 2 *excluding Assumption* 1. *Then for* $\hat{Q} = K' \cap S_r(0)$, *where* $K' = U^0 \cup \{q^i\}$, *the inequality* (5) *is satisfied.*

An analogous result yields in the case where instead of (21) the estimate

$$\|u\|_2^2 \leq c_1\|f\|_0^2 + c_2 \tag{27}$$

holds with c_1, c_2 independent of f and u.

The validity of the estimates (21) and (27) is established for concrete variational inequalities (in particular, cf. Ficchera, 1972 for Signorini problems; Kustova, 1985 for the isotropic variant of contact problems in the elasticity theory).

Regarding the verification of condition (4) for the case that inclusion $K_i \subset K_0$ is not guaranteed, we must remark that this essentially depends on the given expression of K_0 and the used triangulation method for the approximation of the domain. For instance, in case of a problem with an obstacle on the boundary (cf. Hlavacek *et al.*, 1986), where

$$K_0 = \{u \in H^1 : u \geq \omega_0 \quad \text{a.e. on } \Gamma\}, \tag{28}$$

the inclusion $K_i \subset K_0$, generally speaking, cannot be guaranteed. However, if $K_i \subset K_0$ holds for $\omega_0 = 0$ (this happens if Ω is a polyhedron or, in more general situations, by using special approximations on the basis of triangulation with curve-elements near the boundary of Ω), a corresponding inequality can be proved in case where ω_0 is the trace on Γ of any function $\omega \in H^2$. Then any element $v^i \in K_i$ can be expressed by

$$v^i = \omega_{h_i} + s^i, \quad \text{where } s^i \in K'' \equiv K_0 - \omega .$$

Because of $\|\omega - \omega_h\| \leq c\|\omega\|_2 h$, it follows that

$$\rho(K_i, K_0) \leq c\|\omega\|_2 H .$$

Concerning convex semi-infinite programs we consider $V \equiv \mathbb{R}^n$ and K_0 has the expression

$$K_0 = \{u \in U : g(u,t) \leq 0, t \in M\}, \quad U \subset \mathbb{R}^n .$$

Under the usual assumptions (cf. Tichatschke, 1985) that U is convex and closed, M is a compact subset of any normed space Y and, J and g_t $(t \in M): u \to g(u,t)$ are convex, finite valued functions on $\mathbb{R}^n$, the family $\{K_i\}$ is determined by a discretization of M. If $M_i \subset M$ is a finite h_i-grid on the compact M, then the approximated set K_i is given by

$$K_i = \{u \in U : g(u,t) \leq 0, t \in M_i\} .$$

The choice of the corresponding function φ is characterized in the following statement.

THEOREM 4. *Let r be such that $U^0 \cap S_r(0) \neq \emptyset$ and int $S_r(0)$ contains a Slater point $z: z \in U$, $\sup_{t\in M} g(z, t) < 0$. Furthermore, suppose that there exists a constant L such that for every $t, t' \in M$ the inequality*

$$\sup_{u\in U\cap S_r(0)} |g(u, t) - g(u, t')| \leq L\|t - t'\|_Y \tag{29}$$

holds. Then

$$\rho_H(Q_0, Q_i) \leq \frac{2}{a} rLh_i, \quad \text{with } a = -\max_{t\in M} g(z, t),$$

i.e., the conditions (4), (5) *are satisfied with* $\varphi(h) = \frac{2}{a} rLh$.

In the first of the following two examples the initial program is solvable, but the norms of the solutions $u^{*,i} \in \operatorname{Arg\,min}_{u\in K_i} J(u)$ of the auxiliary programs tend to infinity. In the second example in case of the solvability of the initial problem, the discretized problems are unsolvable. But the described approach of iterative regularization guarantees in a natural manner that always the constructed minimizing sequence converges to some element of the solution sets.

EXAMPLE 1. Consider the following simple linear semi-infinite program: $V = \mathbb{R}^2$,

$$U = \{(v_1, v_2): v_2 \geq 0\}, \quad J(u) = -u_1, \quad M = [0, 1],$$
$$g(u, t) = \max\{u_1 - t, u_1 - t^2 u_2\}.$$

For M_i a finite h_i-grid on $[0, 1]$ is chosen such that $t_i^0 \neq 0$, where $t_i^0 = \min\{t: t \in M_i\}$.

It is clear that the points $u^* = (0, a)$, with $a \geq 0$, are the solutions of this program. Considering the discretized problems by replacing M_i instead of M, we get the following solution points $u^{*,i} = (t_i^0, b_i)$, where $b_i \geq (t_i^0)^{-1}$, i.e., $\lim\|u^{*,i}\| = \infty$. Setting $\hat{u}^i = ((t_i^0)^2, 1)$, $r \geq 2$, we have $\hat{u}^i \in K_i \cap S_r(0)$,

$$J(v^i) - J(u^{*,i}) \leq h_i \quad \text{and} \quad \rho(Q_i, Q_0) \leq \min\{h_i, h_i^2 r\}.$$

Hence, for a corresponding choice of $\{h_i\}$, $\{\varepsilon_i\}$ and r the conditions of Theorem 2 are fulfilled and guarantee the convergence of a minimizing sequence $\{u^i\}$ constructed by method (2), (3) to a solution of the initial problem.

EXAMPLE 2. It arises from the first one by replacing g by $g(u, t) = u_1 - tu_2$. Here, $u^* = (0, a)$, $a \geq 0$ and with the same discretization of M we get problems for which $\inf_{u\in K_i} J(u) = -\infty$. Moreover, for this example Assumption 2 cannot be fulfilled. But since $\rho(Q_i, Q_0) \leq rh_i$, the convergence of the modified method with the objective functional (14) is achieved, using the corresponding parameters $\{\varepsilon_i\}$, $\{h_i\}$ and $\{\delta_i\}$.

REMARK 4. The non-constructive condition (2) can be satisfied, for instance, by using the two side functional estimates for the solution of the approximated problems. Such estimates occur in a series of penalty methods and methods of modified Lagrange functions (cf. Grossmann and Kaplan, 1979). Moreover, in case of treating linear or quadratic semiinfinite programs and for the majority of applied variational inequalities we get stable, finite dimensional linear or quadratic programs for which exact solution methods are known.

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